MATHEMATICS 218
SPRING SEMESTER 2008-09
FINAL EXAMINATION-SAMPLE SOLUTION

Time: 120 minutes　　　　Date: June 3, 2009

Name:————————————————————-

ID Number:——————————————————

Circle your section number in the table below:

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Section	7	3	11	1	10	6	2	4
Section		5		12		8	9	

QUESTION	GRADE
1	/25
2	/20
3	/20
4	/20
5	/20
6	/20
7	/20
8	/25
9	/30
TOTAL GRADE	/200

Answer the following nine sets of questions on the allocated pages; the back of pages may be used if needed

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ the transformation defined by

$$T(x, y, z, w) = (x + y - z + w, 2x + y + 4z + w, 3x + y + 9z).$$

(a) Show that T is a linear transformation. (4 points)

Solution. T is a linear transformation since $x+y-z+w$, $2x+y+4z+w$, and $3x + y + 9z$ are linear functions in x. y, and z.

(b) Find the standard matrix $[T]$. (5 points)

Solution.

$$[T] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix}.$$

(c) Find bases for the kernel of and range of T. (12 points)

Solution. We have:

$$\begin{aligned} [T] &= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 4 & 1 \\ 3 & 1 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 6 & -1 \\ 0 & -2 & 12 & -3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 6 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Thus the kernel of T consists of all vectors (x, y, z, w), where $x = -5s, \quad y = 6s, \quad z = s, \quad w = 0$, or,

$$\text{kernel}(T) = \left\{ \begin{bmatrix} -5s \\ 6s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} s & \begin{matrix} -5 \\ 6 \\ 1 \\ 0 \end{matrix} \end{bmatrix} : s \in \mathbb{R} \right\};$$

Hence a basis for kernel(T) is the singleton

$$\left\{ \begin{bmatrix} -5 \\ 6 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

On the other hand, a basis for the range of T consists of the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(d) Find the rank and nullity of T. (4 points)

Solution. It follows from (c) that rank (T)= 3 nullity (T)= 1.

2. Let $T : P_2 \to \mathbb{R}^3$ be the function defined by the formula

$$T(\mathbf{p}(x)) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \\ \mathbf{p}(3) \end{bmatrix};$$

here P_2 is the vector space of all real polynomials of degree at most 2.

(a) Show that T is a linear transformation. (8 points)

Solution. Since for any $\mathbf{p}, \mathbf{q} \in P_2$ and $k \in \mathbb{R}$,

$$T(\mathbf{p}(x)+\mathbf{q}(x)) = \begin{bmatrix} \mathbf{p}(1)+\mathbf{q}(1) \\ \mathbf{p}(2)+\mathbf{q}(2) \\ \mathbf{p}(3)+\mathbf{3}(3) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \\ \mathbf{p}(3) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(1) \\ \mathbf{q}(2) \\ \mathbf{q}(3) \end{bmatrix} = T(\mathbf{p}(x))+T(\mathbf{q}(x))$$

and

$$T(k\mathbf{p}(x)) = \begin{bmatrix} k\mathbf{p}(1) \\ k\mathbf{p}(2) \\ k\mathbf{p}(3) \end{bmatrix} = k \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \\ \mathbf{p}(3) \end{bmatrix} = kT(k\mathbf{p}(x)),$$

T is a linear transformation.

(b) Show that T is one-to-one. (8 points)

Solution. We show that $\mathrm{kerT} = \{0\}$. Note that the general form of a polynomial in P_2 is $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Suppose

$$T(ax^2+bx+c) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} a+b+c \\ a+2b+4c \\ a+3b+4c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This yields the linear homogeneous system of equations

$$\begin{aligned} a+b+c &= 0 \\ a+2b+4c &= 0 \\ a+3b+9c &= 0 \end{aligned}$$

Since

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} = 2 \neq 0$$

the system admits only the trivial solution $a = b = c = 0$; hence $\mathrm{kerT} = \{0\}$ and T is one-to-one.

(c) Show that T is onto. (4 points)

Solution. Since $\mathrm{dimP_2} = \mathrm{dim}\mathbb{R}^3 = 3$ and T is one-to-one linear transformation, T is also onto.

3. Let P_3 be the set of all polynomials of degree at most 3, and let

$$W = \{ax^3 + bx^2 + cx + d : b + c + d = 0\}.$$

(a) Show that W is a subspace of P_3. (6 points)

Solution. Let $a_1x^3 + b_1x^2 + c_1x + d_1, a_2x^3 + b_2x^2 + c_2x + d_2 \in W$. Then $b_1 + c_1 + d_1 = b_2 + c_2 + d_2 = 0$; consequently, $b_1 + b_2 + c_1 + c_2 + d_1 + d_2 = 0$. This implies that

$$\begin{aligned}(a_1x^3 + b_1x^2 + c_1x + d_1) &+ (a_2x^3 + b_2x^2 + c_2x + d_2) \\ &= (a_1 + a_2)x^3 + (b_1 + b_2)x^2 \\ &+ (c_1 + c_2)x + (d_1 + d_2) \in W.\end{aligned}$$

Also, for any real k,

$$k(a_1x^3 + b_1x^2 + c_1x + d_1) = ka_1x^3 + kb_1x^2 + kc_1x + kd_1 \in W$$

since $kb_1 + kc_1 + kd_1 = k(b_1 + c_1 + d_1) = 0$.

(b) Find a basis S for W. (6 points)

Solution. We may write

$$W = \{ax^3 + bx^2 + cx - b - c = ax^3 + b(x^2 - 1) + c(x - 1) : a, b, c \in \mathbb{R}.\}$$

Hence,

$$P_3 = \text{span}\{x^3, x^2 - 1, x - 1.\}$$

But $\{x^3, x^2 - 1, x - 1\}$ is linearly independent since

$$ax^3 + b(x^2 - 1) + c(x - 1) \equiv 0$$

means

$$ax^3 + bx^2 + cx - b - c \equiv 0$$

and consequently $a = b = c = 0$. Hence $S = \{x^3, x^2 - 1, x - 1\}$.

(c) Give one vector in P_3 but not in W. (2 points)

Solution. Any polynomial $ax^3 + bx^2 + cx + d$, where $b + c + d \neq 0$ is not in W; for instance the polynomial $x^3 + x^2 + x + 1$.

(d) Complete S to a basis for P_3. (6 points)

Solution. We show that $B = \{x^3, x^2 - 1, x - 1, 1\}$ is a basis for P_3. Since P_3 has dimension 4, it suffices to show that B is linearly independent. This holds since

$$ax^3 + b(x^2 - 1) + c(x - 1) + d.1 \equiv 0$$

means

$$ax^3 + bx^2 + cx - b - c + d \equiv 0,$$

and consequently $a = b = c = d = 0$.

4. Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

(a) Find a basis for the row space of A and a basis for its orthogonal complement. (16 points)

Solution: (a) Since

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix} \equiv \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

a basis for the row space is $\{(1,3,5,7),(0,1,1,2)\}$.

Since the orthogonal complement of the row space of A is the nullspace of A, the latter consists of all the vectors (x_1, x_2, x_3, x_4) where

$$x_1 = -2s - t,\ x_2 = -s - 2t,\ x_3 = s,\ x_4 = t,\quad s,t \in \mathbb{R}.$$

Thus the nullspace of A, or, the orthogonal complement of the row space of A is the set of all vectors

$$\left\{ \begin{bmatrix} -2s - t \\ -s - 2t \\ s \\ t \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ s \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} : s,t \in \mathbb{R} \right\}$$
$$= \operatorname{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a subset of the column vectors of A that forms a basis for the column space of A. (4 points)

Solution. The desired set is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}. \right\}$$

5. Let P_2 be the set of all polynomials of degree at most 2, and let $\mathbf{p}(x) = a_0 + a_1x + a_2x^2$ and $\mathbf{q}(x) = b_0 + b_1x + b_2x^2$ be in P_2. Define on P_2 the operation

$$< \mathbf{p}, \mathbf{q} >= a_0b_0 + 2a_1b_1 + 3a_2b_2.$$

(a) Show that $< ., . >$ is an inner product on P_2. (10 points)

Solution. Let $\mathbf{r}(x) = c_0 + c_1x + c_2x^2$. Then $< ., . >$ is an inner product on P_2 since the following hold:

$$< \mathbf{p}, \mathbf{q} >= a_0b_0 + 2a_1b_1 + 3a_2b_2 = b_0a_0 + 2b_1a_1 + 3b_2a_2 =< \mathbf{q}, \mathbf{p} >,$$

$$\begin{aligned} < \mathbf{p} + \mathbf{q}, \mathbf{r} > &= (a_0 + b_0)c_0 + 2(a_1 + b_1)c_1 + 3(a_2 + b_2)c_2 \\ &= a_0c_0 + b_0c_0 + 2a_1c_1 + 2b_1c_1 + 3a_2c_2 + 3b_2c_2 \\ &= (a_0c_0 + 2a_1c_1 + 3a_2c_2) + (b_0c_0 + 2b_1c_1 + 3b_2c_2) \\ &= < \mathbf{p}, \mathbf{r} > + < \mathbf{q}, \mathbf{r} >, \end{aligned}$$

$< k\mathbf{p}, \mathbf{q} >= ka_0b_0 + 2ka_1b_1 + 3ka_2b_2 = k(a_0b_0 + 2a_1b_1 + 3a_2b_2) = k < \mathbf{p}, \mathbf{q} >$ for every $k \in \mathbb{R}$, and

$$< \mathbf{p}, \mathbf{p} >= a_0^2 + 2a_1^2 + 3a_2^2 \geq 0$$

with equality if and only if $a_0 = a_1 = a_2 = 0$, or $\mathbf{p} = 0$.

(b) Show that

$$(a_0b_0 + 2a_1b_1 + 3a_2b_2)^2 \leq (a_0^2 + 2a_1^2 + 3a_2^2)(b_0^2 + 2b_1^2 + 3b_2^2).$$

(5 points)

Solution. By the Cauchy-Schwarz inequality we have $| < \mathbf{p}, \mathbf{q} > |^2 \leq \|\mathbf{p}\|^2\|\mathbf{q}\|^2$, or the desired inequality.

(c) Determine the cosine of the angle between the polynomials

$$1 - x + x^2 \quad \text{and} \quad 1 + x - x^2.$$

(5 points)

Solution. Since

$$< 1 - x + x^2, 1 + x - x^2 >= (1)(1) + 2(-1)(1) + 3(1)(-1) = -4,$$

$\|1-x+x^2\| = [1+2(-1)^2+3(1)^2]^{1/2} = 6^{1/2}$ and $\|1+x-x^2\| = [1+2(1)^2+3(-1)^2]^{1/2} = 6^{1/2}$, we have,

$$\cos(1-x+x^2, 1+x-x^2) = \frac{< 1 - x + x^2, 1 + x - x^2 >}{\|1 - x + x^2\|\|1 + x - x^2\|} = \frac{-4}{6^{1/2}6^{1/2}} = \frac{-2}{3}.$$

6. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Use the Gram-Schmidt process to transform the column vectors of A to an orthonormal basis of $\mathbb{R}^3$. (16 points)

Solution: Let $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 2)$, $\mathbf{u}_3 = (2, 1, 0)$. Take $\mathbf{v}_1 = \mathbf{u}_1$. Define

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u}_2 - \frac{< \mathbf{u}_2, \mathbf{v}_1 >}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ &= (0, 1, 2) - (1, 0, 1) \\ &= (-1, 1, 1). \end{aligned}$$

Also define,

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \left\{ \frac{< \mathbf{u}_3, \mathbf{v}_1 >}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{< \mathbf{u}_3, \mathbf{v}_2 >}{\|\mathbf{v}_1 2\|^2}\mathbf{v}_2 \right\} \\ &= (2, 1, 0) - \{(1, 0, 1) + \frac{1}{3}(1, -1, 1)\} \\ &= (2/3, 4/3, -2/3). \end{aligned}$$

Thus an orthonormal system resulting from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ consists of $q_1 = \mathbf{v}_1/\|\mathbf{v}_1\| = (1/\sqrt{2}, 0, 1/\sqrt{2})$, $q_2 = \mathbf{v}_2/\|\mathbf{v}_2\| = (-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $q_3 = \mathbf{v}_3/\|\mathbf{v}_3\| = (1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$.

(b) Find the QR-decomposition of A. (4 points)

Solution. (b) The QR-decomposition of A is

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -\sqrt{3}/3 \\ 0 & 0 & 2\sqrt{6}/3 \end{bmatrix}.$$

7. Let

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

(a) Find the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$. (16 points)

Solution. The normal equation of the linear system $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} -2 & 1 & 1 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 6 & -9 \\ -9 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

But the matrix on the left-hand side is invertible; hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 14 & 9 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

and the least squares solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13/3 \\ 3 \end{bmatrix}.$$

(b) Find the orthogonal projection of $\mathbf{b}$ on the column space of A. (4 points)

Solution. The orthogonal projection of $\mathbf{b}$ on the column space of A is given by

$$A \begin{bmatrix} 13/3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 13/3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -5/3 \\ 4/3 \end{bmatrix}.$$

8. Let

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A. (5 points)

Solution. The CE of A is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} = (\lambda + 1)^2(\lambda - 2) = 0.$$

Hence the eigenvalues are $\lambda_1 = -1$, with algebraic multiplicity 2, and $\lambda_2 = 2$, with algebraic multiplicity 1.

(b) Show that A is diagonalizable. (10 points)

Solution. We find the eigenspaces of λ_1 and λ_2. For the first, consider

$$\lambda_1 I - A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\text{nullspace}(\lambda_1 I - A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

In a similar manner we find

$$\text{nullspace}(\lambda_2 I - A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Since the set of vectors

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is linearly independent, the matrix A is diagonalizable.

(c) Find a matrix P that diagonalizes A and determine $P^{-1}AP$. (5 points)

Solution. a matrix P that diagonalizes A is

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(d) Find A^{10} (5 points)

Solution. We have

$$\begin{aligned} A &= P\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1} \\ &= \end{aligned}$$

9. Indicate whether each of the following statements is TRUE (T) or FALSE (F) without justifying your answer. (3 points each)

—-(a) If A is an $n \times n$ matrix that satisfies $AA^T = I$, then $\det(A) = 1$.

—-(b) If $A^2 = A$ and λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

—-(c) If A is an $n \times n$ matrix invertible matrix, then the orthogonal complement of its nullspace is $\mathbb{R}^n$.

—-(d) A square matrix is diagonalizable if and only if $\lambda = 0$ is an eigenvalue.

—-(e) Any linear system $A\mathbf{x} = \mathbf{b}$ satisfies $\text{rank}[A|\mathbf{b}] = \text{rank}(A)$.

—-(f) Any matrix A can be expressed as a product of elementary matrices.

—-(g) If dim V < dim W < ∞, then there exists a one-to-one linear transformation $T : V \to W$.

—-(h) If a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ satisfies $T(2, -1) \neq 0$, then it is onto.

—-(i) The dimension of the vector space of 3×3 matrices is 10.

—-(j) If A is an $m \times n$ matrix, then $A^T A$ is invertible if and only if the set of column vectors of A is linearly independent.